The [illegible]
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has unit speed

Ex. Reparamaterize $f(t) = \langle 3\sin(t), 2t, 3\cos(t) \rangle$ by arc length [illegible]

Sol: First, we compute the arc length of the function

$$s(t) = \int_0^t |\vec{r}\,'(q)|\, dq \qquad \vec{r}\,'(t) = \langle 3\cos(t), 2, -3\sin(t) \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{9\cos^2(t) + 4 + 9\sin^2(t)} = \sqrt{9+4} = \sqrt{13}$$

$$= \int_{q=0}^{t} \sqrt{13}\, dq$$

$$= \left|\sqrt{13}\,t\right|_0^t = \sqrt{13}\,t - 0 = \sqrt{13}\,t$$

So $s(t) = \sqrt{13}\,t$

$t = \frac{s}{\sqrt{13}}$

Finally, our reparamaterized function is $\vec{p}(s) = \vec{r}(t(s))$

$= \langle 3\sin(\frac{s}{\sqrt{13}}), \frac{2s}{\sqrt{13}}, 3\cos(\frac{s}{\sqrt{13}}) \rangle$

NB: For the curve above, $\vec{p}\,'(s) = \langle \frac{3}{\sqrt{13}}\cos(\frac{s}{\sqrt{13}}), \frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}\sin(\frac{s}{\sqrt{13}}) \rangle$

$\therefore$ the magnitude of $\vec{p}\,'(s) = \sqrt{\left(\frac{3}{\sqrt{13}}\cos(\frac{s}{\sqrt{13}})\right)^2 + \left(\frac{2}{\sqrt{13}}\right)^2 + \left(-\frac{3}{\sqrt{13}}\sin(\frac{s}{\sqrt{13}})\right)^2}$

$= \sqrt{\frac{9}{13}\cos^2(\frac{s}{\sqrt{13}}) + \frac{4}{13} + \frac{9}{13}\sin^2(\frac{s}{\sqrt{13}})}$

$= \sqrt{\frac{9}{13} + \frac{4}{13}} = \sqrt{\frac{13}{13}} = 1$ for all s

Hence, this reparameterized curve has unit speed. In general, a curve parameterized by arc length always has unit speed

Now for some physics

Ex. Find the velocity and acceleration of $\vec{r}(t) = \langle 2^t, t^2, \ln(1+t) \rangle$ at $t=1$

$2^t = e^{\ln(2)t}$

Sol: $\vec{v}(t) = \vec{r}\,'(t)$

$= \langle \ln(2) e^{\ln(2)t}, 2t, \frac{1}{1+t} \rangle = \langle \ln(2) 2^t, 2t, \frac{1}{1+t} \rangle$

So at $t=1$, $\vec{v} = \langle \ln(2) 2^1, 2(1), \frac{1}{1+1} \rangle$

$= \langle 2\ln(2), 2, \frac{1}{2} \rangle$

$\vec{a}(t) = \vec{r}\,''(t) = \vec{v}\,'(t) = \langle \ln(2)^2 e^{\ln(2)t}, 2, -\frac{1}{(1+t)^2} \rangle$

$= \langle \ln(2)^2 2^t, 2, -(1+t)^{-2} \rangle$

$\vec{a}(1) = \langle \ln(2)^2 (2(1)), 2, -(1+1)^{-2} \rangle$

$= \langle 2\ln(2)^2, 2, -\frac{1}{4} \rangle$

Ex. Find the velocity and position functions of the curve with $\vec{a}(t) = \langle \sin(t), 2\cos(t), 6t \rangle$ and $\vec{v}(0) = \langle 0, 0, -1 \rangle$, $\vec{r}(0) = \langle 0, -1, -4 \rangle$

Sol: $\vec{v}(t) = \int \vec{a}(t)\,dt$

$= \langle -\cos(t), 2\sin(t), 3t^2 \rangle + \vec{c}$

Now $\langle 0,0,-1 \rangle = \vec{v}(0) = \langle -\cos 0, 2\sin 0, 3(0)^2 \rangle + \vec{c}$

$= \langle -1, 0, 0 \rangle + \vec{c}$

$\therefore \vec{c} = \langle 0,0,-1 \rangle - \langle -1,0,0 \rangle = \langle 1, 0, -1 \rangle$

$\therefore \vec{v} = \langle -\cos(t), 2\sin(t), 3t^2 \rangle + \langle 1, 0, -1 \rangle$

$= \langle 1-\cos(t), 2\sin(2), 3t^2 - 1 \rangle$

Now $\vec{r}(t) = \int \vec{v}(t)\,dt$

$= \langle t - \sin(t), -2\cos(t), t^3 - t \rangle + \vec{c}$

$\vec{r}(0) = \langle 0 - \sin 0, -2\cos 0, 0^3 - 0 \rangle + \vec{c}$

$\langle 0, 1, -4 \rangle = \langle 0, -2, 0 \rangle + \vec{c}$

$\vec{c} = \langle 0, 3, -4 \rangle$

$\vec{r}(t) = \langle t - \sin(t), 3 - \cos(t), t^3 - t - 4 \rangle$

Ex. When is the speed of particle with position function $\vec{r}(t) = \langle t^2, 5t, t^2-16t \rangle$ at a minimum?

Sol: $\vec{r}\,'(t) = \langle 2t, 5, 2t-16 \rangle$

$f(t) = |\vec{r}\,'(t)| = \sqrt{(2t)^2 + 5^2 + (2t-16)^2}$

$f(t) = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256}$

$= (8t^2 - 64t + 281)^{1/2}$

$\therefore f'(t) = \frac{1}{2}(8t^2 - 64t + 281)^{-1/2}(16t - 64)$

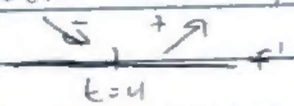
$= \dfrac{8t-32}{(8t^2-64t+281)^{1/2}}$

Note that $64^2 - 4\cdot 8 \cdot 281 = 2^{12} - 2^5 \cdot 281 \le 2^{12} - 2^5 \cdot 256 = 2^{12} - 2^5 \cdot 2^8 = 2^{12} - 2^{13} < 0$

$\therefore 8t^2 - 64t + 281 = 0$ has no real solutions

$\therefore$ the only critical point of this function is at $8t - 32 = 0$ i.e. $t = 4$

Now applying the 1st deriv test, if $f'(t) < 0$ on $t < 4$ and $f'(t) > 0$ on $t > 4$, then $t = 4$ corresponds to a minimum



Now $f'(0) = \frac{-32}{\sqrt{281}} < 0$ and $f'(5) = \frac{+}{(+)} > 0$

Hence the particle is slowest at $t = 4$

Recall: If $f(t) \geq 0$ for all t and f is diff. for all t, then f is minimized exactly when $(f(t))^2$ is minimized

Alt. Sol: $f(t) = |\vec{r}\,'(t)| = (8t^2 - 64t + 281)^{1/2}$ as before

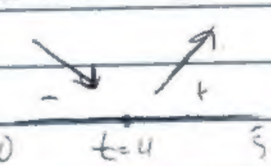
Now we minmize $(f(t))^2 = 8t^2 - 64t + 281$

As before, $8t^2 - 64t + 281 \neq 0$ for all t

Now $(f(t))^2 = g(t)$ is minmized via the 1st deriv test

$g'(t) = 16t - 64$

$g'(t) = 0$ if $t = 4$



$\therefore$ the particle speed is minimized at $t = 4$

Ex. A ball is kicked at angle 60° above ground. If it lands 90m away, at what speed was it thrown? Accel due to grav = 9.8 m/s²

Sol:

$a(t) = \langle 0, -9.8 \rangle$

$\vec{v}(0) = |\vec{v}(0)| \langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \rangle = c\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

$\vec{r}(t_0) = \langle 90, 0 \rangle$

$\therefore \vec{v}(t) = \int \vec{a}(t)\,dt$

$= \langle \alpha, -9.8t + \beta \rangle$

$\vec{v}(0) = \frac{c}{2}\langle 1, \sqrt{3} \rangle$

$\therefore \begin{cases} \alpha = \frac{c}{2} \\ \beta = \frac{\sqrt{3}}{2}c \end{cases}$ $\quad \therefore v(t) = \langle \frac{c}{2}, -9.8t + \frac{\sqrt{3}}{2}tc \rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt$

$= \langle \frac{c}{2}t + \gamma, -4.9t^2 + \frac{\sqrt{3}}{2}c + \delta \rangle$

Now at some time t_0 we have

$\vec{r}(t_0) = \langle 90, 0 \rangle = \langle \frac{c}{2}t_0 + \gamma, -4.9t_0^2 + \frac{\sqrt{3}}{2}ct_0 + \delta \rangle$

We may assume $\vec{r}(0) = \langle 0, 0 \rangle$

Now notice with our assumption $\vec{r}(0) = \langle 0, 0 \rangle$ we obtain $\langle \gamma, \delta \rangle = 0$

$r(t_0) = \langle \frac{c}{2}t_0, -4.9t_0^2 + \frac{\sqrt{3}}{2}c \rangle$ $\quad \therefore \frac{c}{2}t_0 = 90$ so $t_0 = \frac{180}{c}$

$\therefore -4.9\left(\frac{180}{c}\right)^2 + \frac{\sqrt{3}}{2}c = 0$

Ex. A ball is kicked at an angle [illegible] [illegible] it lands 90m away, at what speed was it thrown [illegible]

Sol.

$a(t) = \langle 0, -9.8 \rangle$

$v(0) = |\vec{v}(0)| \langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \rangle = c \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

$r(t_0) = \langle 90, 0 \rangle$

$\therefore \vec{v}(t) = \int \vec{a}(t)\,dt$

$= \langle \alpha, -9.8t + \beta \rangle$

$\vec{v}(0) = \frac{c}{2} \langle 1, \sqrt{3} \rangle$

$\therefore \begin{cases} \alpha = \frac{c}{2} \\ \beta = \frac{\sqrt{3}}{2}c \end{cases}$ $\quad \therefore v(t) = \langle \frac{c}{2}, -9.8t + \frac{\sqrt{3}}{2}c \rangle$

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$r(t_0) = \langle \frac{c}{2}t_0, -4.9t_0^2 + \frac{\sqrt{3}}{2}c \rangle \quad \therefore \frac{c}{2}t_0 = 90$ so $t_0 = \frac{180}{c}$

$\therefore -4.9\left(\frac{180}{c}\right)^2 + \frac{\sqrt{3}}{2}c = 0$